

Capacitance and Dielectrics

George Gamow was a famous nuclear physicist and cosmologist who once wrote an entertaining and humorous series of books for the layman in which many of the mysteries of modern science were explained in simple terms. In one of these books Professor Gamow introduced a magical character termed a Maxwellian Demon who had the ability to perform everyday chores on an atomic scale. We will make use of such a character at times in this article while affectionately referring to him as Max. Max is quite industrious and has brought about the four situations depicted in Fig. 1.

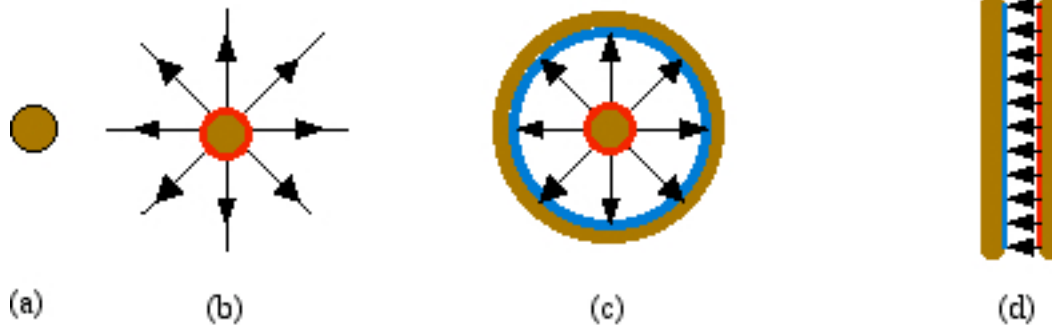


Figure 1. The Maxwellian Demon's initial handiwork.

Fig. 1 (a) represents a two dimensional cross-section of an isolated copper sphere where 'isolated' means that the sphere is far removed from any other objects. In Fig. 1 (b), Max has removed a quantity of negative charge from the copper sphere and distributed it at an infinitely remote distance thus leaving the sphere with an excess positive charge of amount q . Following Max's removal of the quantity of negative charge from the sphere the remaining charges in the sphere will redistribute themselves until electrostatic equilibrium is again restored at which time the excess positive charge will all reside on the outer surface of the sphere with a uniform surface density. This is represented in the two dimensional figure by a red circle surrounding the copper core. We know from Gauss's law there will now exist a radially directed electrostatic field in the space surrounding the sphere with this field originating on the positive surface charge. We also know that the interior electrostatic field is zero as there is no excess charge in the interior of a conductor under electrostatic equilibrium. The absence of the field in the interior means that there is no potential difference between interior points so all such points must have the same value for the potential and that the entire sphere must be an equipotential body. We can calculate the value of this potential possessed by the sphere through the employment of the technique presented in the preceding article in this series of articles.

$V_a = \int_a^{\infty} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \int_a^{\infty} \frac{q}{r^2} dr = \frac{q}{4\pi\epsilon_0 a}$, where a is the radius of the copper sphere. The

capacitance of this isolated, charged conducting sphere is defined to be the ratio of its charge, q , to its resulting potential, V_a , so its capacitance in units of farads is

$C = \frac{q}{V_a} = 4\pi\epsilon_0 a$. In Fig. 1 (c), Max introduces us to a device called a capacitor even

though it is a highly impractical one. A capacitor consists of two conductors that are

normally insulated from each other. In this instance, the first conductor is the original charged copper sphere that is now surrounded by a hollow spherical shell that is centered on the origin of the solid sphere. Additionally the negative charge that Max removed from the solid sphere now resides uniformly on the inner surface of the spherical shell with empty space existing between the two conductors. If b is the inner radius of the spherical shell then the potential difference between the two conductors assumes the

value of $V_{ab} = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{a} - \frac{1}{b} \right] = \frac{q}{4\pi\epsilon_0} \left[\frac{b-a}{ab} \right]$. The capacitance of this capacitor is

$C = \frac{q}{V_{ab}} = 4\pi\epsilon_0 \left[\frac{ab}{b-a} \right]$. In both instances the capacitance is proportional to ϵ_0 multiplied

by a length quantity characterized by the conductor or conductors involved. Finally, in Fig. 1 (d) Max introduces us to the simplest of practical capacitor structures. This is the parallel plate capacitor. It is practical because we can attach a conducting lead to each conductor forming the makeup of the capacitor and thereby connect the capacitor to a circuit. It is the simplest because if the separation between the plates is much smaller than the lateral extent of the capacitor plates, the electric field between the plates will be quite uniform with only a slight fringing very near the edges of the plates. This fringing has a negligible effect on our calculations. Upon neglecting the effect of fringing, the positive charge will be uniformly distributed on the inner surface area of one of the plates while the equal amount of negative charge will be uniformly distributed on the inner surface of the opposite plate. Let the surface area of either plate containing charge be called A and let the separation between the plates be called d . Let the charge that has been transferred between the plates be called q so that the surface density of charge on the positively

charged plate is $\frac{q}{A}$ while that on the negatively charged plate is $-\frac{q}{A}$. The electric field

originates at the positively charged plate and is perpendicular to the positively charged surface. This same field terminates at the negatively charged plate and is perpendicular to the negatively charged surface as indicated in the drawing. The strength of this uniform

field as determined from an application of Gauss's law is $E = \frac{1}{\epsilon_0} \frac{q}{A}$ provided that only

empty space exists in the region between the plates. If we pick a path along a perpendicular drawn from the positive plate to the negative plate near the center of the structure, the potential difference between the plates or the voltage of the capacitor is

calculated as $V = \int_{\text{pos}}^{\text{neg}} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{\epsilon_0} \frac{q}{A} d$, where d is the separation between the plates. By

definition, the capacitance of the structure is the ratio of the transferred charge to the

resulting potential difference or $C = \frac{q}{V} = \epsilon_0 \frac{A}{d}$. The unit of capacitance in the SI system is

the farad, which is a shortened version of Michael Faraday's surname. Faraday pioneered the study of capacitance and is rightly honored in this fashion. He is further honored making full use of his surname in that the Faraday is a quantity of charge equal to

Avogadro's number of electronic charges. Let's analyze the dimensions of $\epsilon_0 \frac{A}{d}$. When

ϵ_0 was first introduced in a previous article dealing with Coulomb's law, it was stated that

$\epsilon_0 = 8.85(10^{-12})$ Coulomb²Newton⁻¹meter⁻². Now area divided by distance has the dimension of a meter so the dimensions become Coulomb²Newton⁻¹meter⁻¹ = Coulomb²Joule⁻¹ = (Coulomb)÷(Joule/Coulomb) = Coulomb/volt = farad. Suppose the area is one meter² and the separation is one mm or 10^{-3} meter. The capacitance would then be only 8.85 nanofarad. In order to achieve an appreciable capacitance, a premium is placed on structures having large plate areas, small plate separations, and perhaps a magic ingredient that Faraday also discovered, namely a dielectric. A dielectric is an insulating material whose properties we will soon discuss. For the moment we are interested in the energy that is stored in the capacitor as the result of the work done by Max in the process of charging the capacitor. Max charges the capacitor by repetitively removing a small increment of negative charge from the right plate in Fig. 1 (d) and subsequently depositing the increment of charge on the left plate in the figure. At any given instant in the process the charge that has been transferred is q' and the resulting potential difference is V' . V' , however, is q'/C so the increment of work performed by Max at each step of the process is $dW = V'dq' = \frac{q'}{C}dq'$ and the total work performed in

the process is given by $W = \frac{1}{C} \int_0^q q'dq' = \frac{1}{2} \frac{q^2}{C}$. This amount of energy is stored in the

capacitor in the form of potential energy that can perform useful work when the capacitor is allowed to discharge. Faraday had the idea that this potential energy was associated with the electric field that exists in the space between the capacitor plates and at a later date Maxwell derived an expression for the electric potential energy density that we can duplicate here for our parallel plate capacitor. From the definition of capacitance we may write $q = CV$ and substitute this expression for q into the equation for the stored energy to

obtain $W = \frac{1}{2}CV^2 = \frac{1}{2}\epsilon_0 \frac{A}{d} V^2 = \frac{1}{2}\epsilon_0 A \frac{V}{d} V = Ad \frac{1}{2}\epsilon_0 \frac{V^2}{d^2} = Ad \frac{1}{2}\epsilon_0 E^2$. Now as Ad is the volume of the space between the plates of the parallel plate capacitor and recognizing that the electric field is uniform in this region, the conclusion is that the potential energy density throughout the volume of Ad must be $\frac{1}{2}\epsilon_0 E^2$.

We now turn to the question of insulators or dielectrics and the roles that they play in electrical capacitors. Dielectric materials may be solid, liquid, or gas while being either purely atomic or molecular in their structure. The solid dielectric structures themselves may be plastic, amorphous, or crystalline. Crystal dielectrics have properties that depend upon the relative direction between the applied electric field and the axes of the crystal whereas non-crystalline dielectrics have isotropic behavior. Most of the dielectrics employed in coaxial cables are chemical compounds and thus are molecular in structure. These molecular structures are classified as being composed of either polar or non-polar molecules. Non-polar molecules have internal charge distributions such that the centroids of the positive charge and the negative charge coincide and thus produce essentially no electric field external to the molecule. In the case of polar molecules, however, the centroids of the positive and negative charge distributions do not coincide so that each molecule possesses an electric dipole moment and produces an external electric field characteristic of an electric dipole. A sample of a dielectric formed from

such molecules will contain an extremely large number of molecules and will not normally display the existence of an electric field because the thermal motion in the dielectric structure produces a random orientation of the electric dipoles resulting in a cancellation of any total overall field. The behavior of both polar as well as non-polar molecules changes when such dielectric structures are subjected to an externally applied electric field as illustrated in Fig. 2.

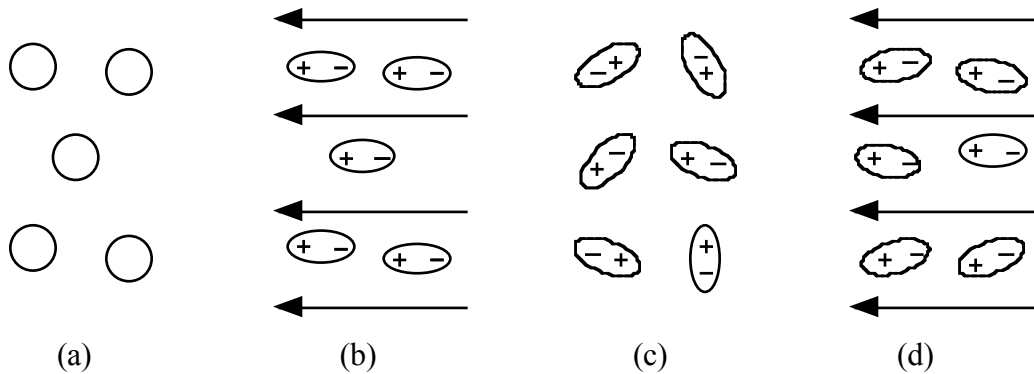


Figure 2. Induced polarization of non-polar and polar molecules.

Fig.2 (a) depicts molecules of a non-polar dielectric in the absence of an externally applied electric field. In this instance the centers of positive and negative charge of each molecule coincide and the molecules produce no electric field of their own. In Fig. 2 (b) an externally applied electric field is present and the centers of positive charge are shifted in the direction of the field while the centers of negative charge are shifted in the opposite direction so that each molecule now has an induced electric dipole moment. This means that each molecule now acts as an electric dipole. The fields of these induced dipoles are in fact oppositely directed to that of the original external field so that the strength of the total resulting electric field is less than that of the original applied field. Fig. 2 (c) is suggestive of the random orientation of polar molecules in the absence of any externally applied electric field. In this instance the vector sum of the individual dipole fields is zero when a large number of molecules is involved, as would be the case in a macroscopic sample of dielectric material. Finally, in Fig. 2 (d) an externally applied electric field produces partial orientation of the polar molecules in a preferred direction with the result that the final total value of the electric field is less than it would have been in the absence of the dielectric material. Given a capacitor with a fixed charge q and no dielectric between the plates such that the original electric field in the region between the plates has a value of say E_0 , the introduction of a dielectric of either the polar or non-polar variety between the plates will result in the electric field taking on a new value $E = \frac{E_0}{\kappa}$, where κ is called the dielectric constant with κ being a dimensionless quantity greater than one. We will now apply these considerations to our parallel plate capacitor and employ Gauss's law to calculate the results.

Fig. 3 illustrates the situation that results after the previously charged parallel plate capacitor with empty space between the plates of Fig. 1 (d) has the empty space filled with a uniform dielectric of constant κ .

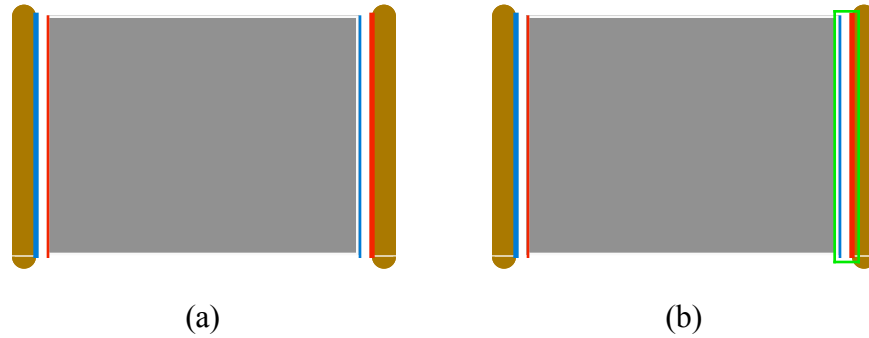


Figure 3. Parallel plate capacitor with dielectric.

In Fig. 3 (a) as well as Fig. 3 (b), the horizontal dimension has been greatly exaggerated in order to clearly display the various charged surfaces. Fig. 3 (a) displays the original charged copper plates of the parallel plate capacitor with the right hand plate bearing a positive charge on its left hand surface as represented by the heavy red line while the left hand plate has an equal negative charge on its right hand surface represented by the heavy blue line. These charges are respectively q and $-q$. These charges are somewhat unique in that they are free to change when the capacitor is either charged or discharged through an external circuit. We will distinguish them by tacking on a subscript and henceforth will identify them as q_f and $-q_f$. In Fig. 3 (a), the dielectric appears as the gray shaded area between the capacitor plates. The right hand surface of the dielectric bears a negative surface charge that is represented by the thin blue line while the left hand surface bears a positive charge represented by the thin red line. These surface charges are the result of the induced dipole moments brought about either the stretching of non-polar molecules or the orientation of polar molecules in the dielectric material. These charges never leave the body of the dielectric and are referred to as being bound charges $-q_b$ and q_b , respectively. Even though the dielectric completely fills the space between the copper plates, the drawing displays a white gap on either side in order to clearly distinguish the surface charge layers on the plates from those on the dielectric. The electric field depends on both free and bound charges and is calculated as follows. In Fig. 3 (b) the green rectangle represents the edge view of an imaginary box that encloses the positive free charge on the copper plate and the negative bound charge on the dielectric surface.

According to Gauss's law the net electric flux directed out through the surfaces of this box is $\frac{(q_f - q_b)}{\epsilon_0}$. None of this flux is directed through the right face of this box as it lies

within the copper plate that is an equipotential body. Similarly there is no flux directed through the top and bottom surfaces of this box as the field must be normal to the vertical surface of the copper plate. Hence the flux must be directed through the dielectric surface

from right to left and the electric field is this flux per unit area or $E = \frac{(q_f - q_b)}{\epsilon_0 A}$. Thus the

electric field in the dielectric is directed from right to left along the horizontal and is uniform in the space between the capacitor plates that is fully occupied by the dielectric.

This new value of the electric field is less than the former value of $\frac{q_f}{\epsilon_0 A}$ that existed prior

to the introduction of the dielectric into the empty space between the capacitor plates.

The total dipole moment of the dielectric is $q_b d$ and is also directed from right to left. The polarization is the dipole moment per unit volume, however, and is $\frac{q_b d}{Ad} = \frac{q_b}{A}$. With the

dielectric in place there are now three vector fields existing in the space between the plates of the capacitor. The first of these is called the displacement symbolized by the vector \mathbf{D} , the second is the polarization or dipole moment per unit volume symbolized by the vector \mathbf{P} , and of course the familiar electric field symbolized by \mathbf{E} . The displacement owes its existence to the free charge on the capacitor plates while the polarization is associated with the bound charge on the dielectric surfaces. The relationship between these three vector fields is then $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. Each of these fields is uniform in value and directed from right to left in our example parallel plate capacitor. As these three fields are parallel in the same direction, the vector equality will also hold for the individual

magnitudes so $D = \epsilon_0 E + P$. We know both E and P so $D = \epsilon_0 \left(\frac{q_f - q_b}{\epsilon_0 A} \right) + \frac{q_b}{A} = \frac{q_f}{A}$. The

final polarization of our isotropic dielectric depends on the final electric field in the dielectric such that $\mathbf{P} = \eta \mathbf{E}$, where η is called the dielectric susceptibility. The dielectric constant κ and the dielectric susceptibility are related in that $\kappa = 1 + \frac{\eta}{\epsilon_0}$. Finally, if we

multiply this last equation by ϵ_0 we obtain a quantity called the electric permittivity of the dielectric given by $\epsilon = \epsilon_0 + \eta = \kappa \epsilon_0$. Upon replacing \mathbf{P} by $\eta \mathbf{E}$ in the displacement equation we obtain $\mathbf{D} = \epsilon_0 \mathbf{E} + \eta \mathbf{E} = \epsilon \mathbf{E}$ and in a vacuum where $\eta = 0$, $\mathbf{D} = \epsilon_0 \mathbf{E}$. Now that we have introduced the displacement field we can write the equation for the potential energy density in its most general field form. The potential energy density in an electric field is

symbolized by U with $U = \frac{1}{2} \mathbf{D} \cdot \mathbf{E}$. Now that all of the tools are in place, a comparison

can be made between the initial and final states of the parallel plate capacitor having a fixed charge q_f . In the initial state the space between the plates is empty while in the final state the space between the plates is filled with a uniform isotropic linear dielectric whose dielectric constant is κ , whose susceptibility is η , and whose permittivity is ϵ . The comparison is displayed in Table 1.

Quantity	Initial	Final
Capacitance	$\epsilon_0 A / d$	$\kappa \epsilon_0 A / d$
Charge	q_f	q_f
Electric Field	$q_f / (\epsilon_0 A)$	$q_f / (\kappa \epsilon_0 A) = (q_f - q_b) / (\epsilon_0 A)$
Polarization	0	q_b / A
Displacement	q_f / A	q_f / A
Voltage	$(q_f d) / (\epsilon_0 A)$	$(q_f d) / (\kappa \epsilon_0 A)$
Stored Potential Energy	$(q_f^2 d) / 2(\epsilon_0 A)$	$(q_f^2 d) / 2(\kappa \epsilon_0 A)$

Table 1. A comparison of conditions existing in the capacitor before and after insertion of the dielectric.

All of the entries in Table 1 have been covered in the foregoing discussion with the possible exception of the final value of the stored potential energy in the capacitor with the dielectric completely filling the space between the plates. Recall that the dielectric constant κ is a number greater than one for a material dielectric. This means that the insertion of the dielectric has reduced the potential energy stored in the system. How can this be so? With the assistance of Max and a perusal of Fig. 4 we can answer this question. A fringing electric field must always exist in the vicinity of the edges of the capacitor plates as is suggested in the simplified drawing of Fig. 4. This is true because the line integral of the electrostatic field around a closed path must be zero. If you pick a path a portion of which is between the plates and the remainder of which is outside of the plates, the contribution to the line integral of an external fringing field is necessary to cancel that of the interior field between the plates.

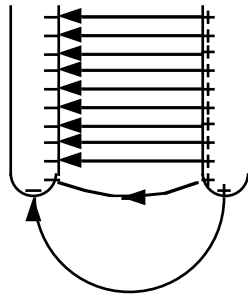


Figure 4. Depiction of a portion of the fringing field near the capacitor plate edges.

Imagine that Max approaches the capacitor from a distance below the capacitor with the dielectric slab centered on the plates. As the dielectric enters the weak fringing electric field it becomes weakly polarized with positive bound charge appearing on its left face and negative bound charge on its right face so that the dielectric is actually attracted to the empty region between the capacitor plates. Max must exert an oppositely directed force in order to prevent the dielectric slab from accelerating and increasing its kinetic energy. In the process the capacitor is doing work on Max and thus the capacitor's stored energy is being decreased. This continues to an increasing degree until the slab is centered in the region between the capacitor plates. If now Max were to remove the dielectric from the region between the plates he would be doing work on the capacitor in restoring its potential energy to the former value.