Mathematics Primer for Vector Fields

By Dr. Eugene Patronis

This is the second in a series of articles dealing with coaxial cables operating in the frequency span from direct current through the microwave region. In order to make the ideas meaningful to the widest audience this article will be devoted to the mathematics necessary to properly describe physical variables that have a dimension or unit, a size or magnitude, and a direction in space.

For the moment we will be concerned with physical variables that fall into two distinctly different mathematical categories. The first category includes items such as mass, charge, density, pressure, and temperature among others. These are scalar quantities that may have a particular unit but otherwise have only a numerical value or magnitude and are subject to the rules of ordinary arithmetic and algebra. The second category includes items such as displacement, velocity, acceleration, force, etc. These are vector quantities that possess a particular unit, a scalar magnitude, and a direction in space as well. Scalar and vector physical variables both may be called field variables if they change as functions of position in space and possibly of time.

Fig. 1 illustrates a graphical technique for adding or subtracting two vectors that is familiar to nearly everyone. What we describe here is a more formal technique for making calculations in a fully developed three-dimensional space. We will employ Cartesian coordinates at the outset, as these are the most familiar while introducing other coordinate systems as the situation dictates. Unit vectors are dimensionless vectors having a scalar magnitude of one and whose sole role is to indicate direction along a coordinate axis. In Cartesian coordinates the unit vectors are usually written $\hat{i}, \hat{j}$, and $\hat{k}$ indicating unit vectors along the x, y, and z-axes, respectively. The hat symbol above each letter symbolizes that the vector is a unit vector having only direction. In the Cartesian coordinate system, as is true in any set of orthogonal coordinates, the unit vectors form a mutually perpendicular set. An ordinary vector having magnitude and dimension as well as direction is often distinguished in printed text by employing a boldface symbol or by an arrow or bar over the chosen symbol. In this article vector quantities will be symbolized by employing a boldface letter symbol. As an example, the general vector force in Cartesian coordinates according to this notational scheme would be written as $\mathbf{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ where F_x is the scalar magnitude of the force along the x-coordinate axis and similarly for F_y and F_z. These individual scalar values may be either positive or negative. Vector quantities having like dimensions may be added or subtracted. For example, consider two vector quantities $\mathbf{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\mathbf{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$ that have the same dimensions or units. These two vectors may be added to produce a new vector say $\mathbf{c} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} + (a_z + b_z)\hat{k}$ with the summation proceeding on a component-by-component basis. Similarly, the vector **b** may be subtracted from **a** to produce a new vector say $\mathbf{d} = (a_x - b_x)\hat{i} + (a_y - b_y)\hat{j} + (a_z - b_z)\hat{k}$. The magnitude of a vector **a** is written as $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$.

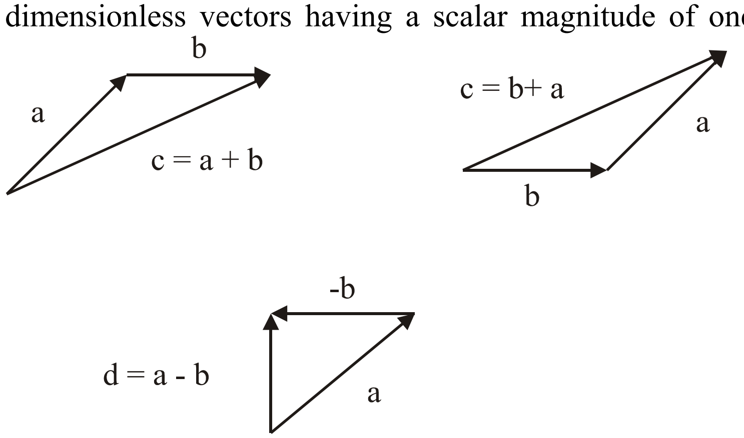
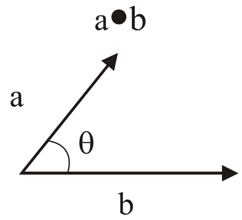


Figure 1. Graphical addition and subtraction of vectors.



a•b = b•a = (a)(b)cos(θ)

Figure 2. Scalar product of two vectors.

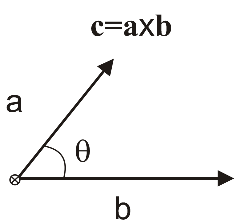
Suppose the vector **r** represents displacement from the origin with $\mathbf{r} = 3\hat{i} + 4\hat{j} + 12\hat{k}$ meters then $|\mathbf{r}| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$ meters as the magnitude of the total displacement from the origin.

Three different methods are defined for products involving vectors. The first is the multiplication of a vector by an ordinary scalar quantity. This is simply a scaling process as 10**F** is a vector in the direction of **F** whose magnitude has been increased by ten fold. The second is that of the scalar product of two vectors. The scalar product of two vectors yields a scalar quantity whose magnitude is the product of the magnitudes of the individual vectors multiplied by the cosine of the angle included between the directions of the individual vectors as shown in Fig.2. Let the two vectors be **a** and **b** as given above. The scalar product of these two vectors is written as $\mathbf{a} \bullet \mathbf{b}$ with this product being given by

$$\begin{aligned}
&\mathbf{a} \bullet \mathbf{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \bullet (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) = \\
&a_x b_x \hat{i} \bullet \hat{i} + a_x b_y \hat{i} \bullet \hat{j} + a_x b_z \hat{i} \bullet \hat{k} + a_y b_x \hat{j} \bullet \hat{i} + a_y b_y \hat{j} \bullet \hat{j} + \\
&a_y b_z \hat{j} \bullet \hat{k} + a_z b_x \hat{k} \bullet \hat{i} + a_z b_y \hat{k} \bullet \hat{j} + a_z b_z \hat{k} \bullet \hat{k} = \\
&a_x b_x + a_y b_y + a_z b_z
\end{aligned}$$

This is true because in the scalar product of a unit vector with itself the included angle is zero for which the cosine of the angle is unity and the scalar product of similar unit vectors is thus unity. On the other hand, the angle between dissimilar unity vectors is 90° for which the cosine of the angle is zero and the scalar product is thus zero. Therefore, $\hat{i} \bullet \hat{i} = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} = 1$ and $\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{i} = \hat{j} \bullet \hat{k} = \hat{k} \bullet \hat{i} = \hat{k} \bullet \hat{j} = 0$ It should be noted that $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$. The process given above has taken the magnitude of **a**, multiplied it by the magnitude of **b**, and then multiplied this product by the cosine of the angle between the two vectors.

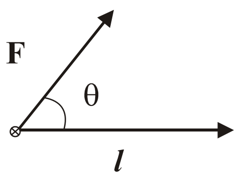
The third method of multiplying two vectors is termed the vector product. The vector product is symbolized by $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ where **c** is the product vector, **a** is the multiplying vector, and **b** is the multiplicand vector. The vector product is itself a vector having a magnitude equal to the product of the magnitudes of the individual vectors further multiplied by the sine of the angle included between the directions of the individual vectors as shown in Fig. 3. Two non-parallel vectors determine the orientation of a plane and the product vector of these two vectors is perpendicular to this plane and bears a right-handed screw relationship with the imagined rotation of the multiplying vector toward the multiplicand vector. This is indicated in Fig.3 by the presence of the screwhead at the origin. As the tip of **a** is rotated towards **b** through the angle θ a right-handed screw would advance away from the reader. As a consequence, $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) = -\mathbf{c}$. If $\mathbf{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\mathbf{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then the product vector **c** may be calculated by direct expansion as was done in the scalar product case with the difference that now one is taking the vector products of the unit vectors in each instance rather than the scalar product as was the former case. Now, of course, $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ while $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, and $\hat{k} \times \hat{i} = \hat{j}$. Further, $\hat{i} \times \hat{k} = -\hat{j}$, $\hat{j} \times \hat{i} = -\hat{k}$, and $\hat{k} \times \hat{j} = -\hat{i}$. Rather than calculating by direct expansion a somewhat speedier calculation is possible by means of the determinant



$|\mathbf{c}| = (a)(b)\sin(\theta)$

Figure 3. Vector product of two vectors.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$



$W = \mathbf{F} \bullet \boldsymbol{l} = Fl\cos(\theta)$

Figure 4. The calculation of work involves a scalar product.

Evaluation of the three by three determinant according to the usual rules yields correctly $\mathbf{c} = (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}$. Note that if **a** and **b** each are displacement vectors then the magnitude of the vector **c** is the area of a parallelogram having **a** and **b** as sides**.** The truly unique property of vectors is the manner in which they transform from one coordinate system to another. This transformation property is such that the vector magnitude and direction remain unchanged under coordinate transformation. This will be explored at the appropriate time.

Imagine that you are exerting a constant force of magnitude F in pushing a loaded wheelbarrow across a horizontal surface through an extended distance along the horizontal of an amount *l*. The calculation of the work done by your applied force in the process is an example of the scalar product of two vectors. The force that you apply has both an upward component because you must lift the handles of the wheelbarrow and a horizontal component in urging the wheelbarrow in the forward horizontal direction. If the vector **F** represents the total force and the vector ***l*** represents the subsequent displacement of the wheelbarrow along the horizontal then the work performed is given by the scalar product contained in $W = \mathbf{F} \bullet \boldsymbol{l}$. This is depicted in Fig. 4. Notice in the figure that multiplying the magnitude of the force by the cosine of the angle θ selects the component of the force that is along the horizontal so that the scalar amount of work is W=F*l* cos(θ). A more realistic example would be one where the force that you have to exert varies dependent on where you are located along the horizontal path. In such an instance we must sum all of the increments of work performed in bringing about the total amount of horizontal displacement. This is done by integration along the horizontal path given by $\int_i^f dW = \int_i^f \mathbf{F} \bullet d\boldsymbol{l}$ where i represents the coordinates of the initial point and f those of the final point defining the horizontal path. Such an integral is called a line integral. An example of the vector product is depicted in Fig. 5 where one desires to calculate the vector force **F** experienced by a length *l* of straight wire conducting a steady current I in a region over which there is a uniform magnetic induction described by the vector **B**. In this instance, a vector **I*l*** spatially describes the current carrying conductor. This vector lies along the length of the conductor and points in the positive sense of the current. The force experienced by the conductor is given by $\mathbf{F} = \mathbf{I}\boldsymbol{l} \times \mathbf{B}$ and has a scalar magnitude F= I*l* Bsin(θ). Note that if the length of the conductor were parallel to the direction of the magnetic induction the angle θ would be zero and there would be no force. Note also from the figure that if the sense of the current were reversed, the direction of the force would reverse also.

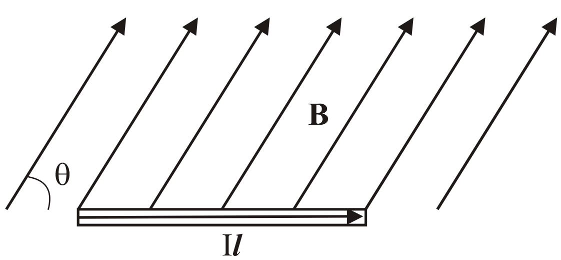


Figure 5. The magnetic force on current carrying conductor involves a vector product with the force on the current element being towards the reader.

Special methods are also required in the calculus of vectors. These methods involve the introduction of a vector differential operator termed del or sometimes nabla that is commonly symbolized by an inverted Greek letter delta written as ∇. In Cartesian coordinates the vector operator del appears as $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ where the indicated derivative operators are partial derivatives. This operator may be applied to both vector and scalar field quantities in a number of ways. Remember a field quantity is a physical variable that varies from point to point in space and may or may not also vary in time. Vector field quantities also have direction whereas scalar field quantities have no direction. As an example of a scalar field quantity suppose we know the temperature distribution in a room where the z-coordinate describes elevation and the x and y coordinates pinpoint location in a horizontal plane at some instant of time. This knowledge is expressed in a mathematical function T. T is a function of the spatial coordinates so we say T= T(x,y,z). With ordinary heating and cooling systems we would expect that the temperature in a plane would not vary dramatically from point to point but that it may vary rather markedly as we vary the elevation of the point of observation. We seek the answer to the following questions. What is the direction of the maximum increase in the function T and what is the magnitude of the rate of increase or slope along this direction? These questions are answered by the gradient of T written as ∇T.

$$\nabla T = \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$$

We see that the gradient of the scalar temperature function is itself a vector. The direction of this vector is that of the maximum rate of change of the function T and the magnitude of this vector is the maximum slope of the function T. This gradient operator will be very important when we study the electric field and the associated scalar electric potential function.

The del operator may be applied to a vector field in two different ways. The first way involves the calculation of the divergence of the vector field quantity. Suppose we have a vector field that is described by $\mathbf{v} = \mathbf{v}(x, y, z) = \hat{i}x + \hat{j}y + \hat{k}z$. This is just a vector displacement field whose magnitude grows linearly with distance from the origin. The divergence of this vector field is written as $\nabla \bullet \mathbf{v}$ and is calculated as

$$\nabla \bullet \mathbf{v} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \bullet (\hat{i}x + \hat{j}y + \hat{k}z) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3\,.$$

This measures the combined growth rates of the individual scalar components along the respective coordinate axes of the vector. $\nabla \bullet \mathbf{v}$ is itself a scalar quantity. The positive divergence of a vector field as is the case here indicates the presence of a source of the field. A negative divergence would be indicative of a sink of the associated vector field.

The second way of applying the del operator to a vector field is through the calculation of the curl of the vector field quantity. The curl of a vector field is itself a vector and measures the space rate of change that occurs at right angles to the direction of the original vector at a given point. Let $\mathbf{v}$ now be $\hat{i}v_x + \hat{j}v_y + \hat{k}v_z$. The curl of $\mathbf{v}$ is written as $\nabla \times \mathbf{v}$ and is calculated as the determinant

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}.$$

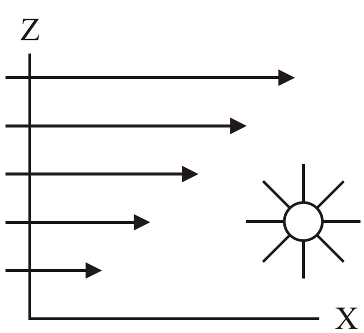


Figure 6. Graphical example of a vector field having curl.

This will be much more meaningful if we examine a physical example that is depicted in Fig. 6. This figure depicts a velocity field in a river as a function of the elevation above the bed of the center of the stream in a region where the river is straight and the riverbanks are parallel. The elevation above the bed is the z-coordinate and the stream flow is along the positive x-coordinate. The y-axis is into the paper and has a constant value of zero for the stream center. As a result of viscous effects the stream velocity is zero on the riverbed and grows linearly with elevation until the surface is met. The velocity field in the region of $0 \le z \le 10$ feet, the depth of the river, is then $\mathbf{v} = \hat{i}az + \hat{j}0 + \hat{k}0$ where *a* is a constant equal to 0.5 (mile hour^{-1}foot^{-1}). Clearly, the stream velocity is in the direction of the x-axis and has a value that only depends on elevation with a maximum value of 5 miles per hour. Upon substitution of $\mathbf{v} = \hat{i}az$ into the determinant expressing the curl of **v** it is found that $\nabla \times \mathbf{v} = \hat{j}a$. Note that the curl of **v** at all points is everywhere perpendicular to **v**. Fig. 6 indicates the presence of a paddle wheel adjacent to the velocity field in the drawing. The axis of the paddle wheel passes into and out of the paper in its present location. This is a useful graphical aid in determining the existence of a curl associated with a vector field. If the paddle wheel were exposed to the vector field as shown the wheel would rotate in a clockwise direction. A right-handed screw rotated in such a manner would advance into the paper along the positive y-axis that is in the direction of the curl. If the axis of the paddle wheel were along either the x-axis or the z-axis, the velocity field would not rotate the wheel.

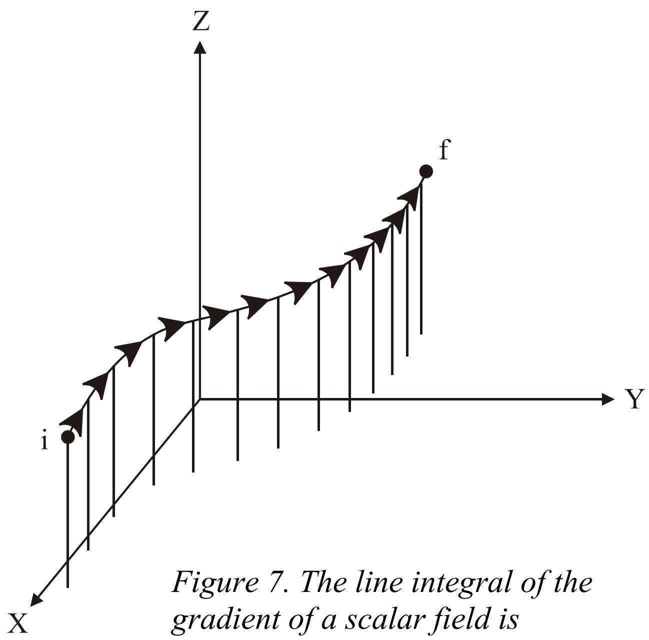


Figure 7. The line integral of the gradient of a scalar field is independent of the path.

We will close this article with one of the fundamental theorems of vector analysis called the gradient theorem. There are two other fundamental theorems associated with the application of the del operator called the divergence theorem and curl theorem that we will encounter next time in connection with learning about the electric field. Fig. 7 depicts a possible path between two points located in a three-dimensional space such as the room mentioned earlier where the scalar temperature field is T = T(x,y,z). The path depicted in Fig. 7 may be described as accurately as you please by differential vector displacements d***l*** each of which is tangent to the actual path at each point in question. Now ∇T is a vector whose magnitude and direction at any point is that of the maximum space rate of change of the temperature at that point. Therefore, the change in temperature when undergoing the differential displacement d***l*** will be given by the scalar product of ∇T with the vector d***l*** or $dT = \nabla T \bullet d\boldsymbol{l}$. The function dT is itself a function of the space coordinates x,y, and z and if we simply require that it be an integrable function then if we integrate dT along the path from the initial point i to the final point f we obtain the result

$$\int_i^f dT = \int_i^f \nabla T \bullet d\boldsymbol{l} = T(f) - T(i).$$

The conclusion indicates that the temperature difference between the ending point and the starting point depends only on the difference between the temperature field function evaluated at the points in question and is independent of the path chosen between the two points in question. Finally, if the initial point and the final point are one and the same point as implied by the circle on the integral symbol below, we may conclude that the integral along a path that closes on itself will be

$$\oint \nabla T \bullet d\boldsymbol{l} = 0.$$