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The Properties of Coaxial Cable

Part 1 - The Magnetic Field

This is the first in a short series of articles that will describe the properties of coaxial cables as deduced from both the vector field theory and circuit theory points of view. Both of these perspectives stem from Maxwell's equations. This first article deals with the magnetic and hence inductive properties of coaxial cables. The electric properties and combined electromagnetic properties will be discussed in a later article.

Initially, the discussion will involve a cable that supports a low frequency sinusoidal alternating current of fixed amplitude. This current is a bonafide signal current and no common mode source is considered to be present. The current in both the inner conductor and the outer conductor have the same magnitude but are oppositely directed. The self-inductance of such a system is a property of the system and is a parameter that quantifies the energy stored in the magnetic field of the system. The relationship between the stored magnetic energy and the current at any instant is $W=1/2\ Li^2$, where W is the instantaneous stored magnetic energy, L is the coefficient of self-inductance and i is the instantaneous value of the current. The structural details are as follows. The inner conductor is a copper wire of radius, a, that is centered on the z-axis. The outer conductor is also of copper and is a hollow cylinder having an inner radius, b, and an outer radius, c. This cylinder is also centered on the z-axis. This system has cylindrical symmetry so cylindrical coordinates will be employed throughout. The region between the two conductors is filled with a solid dielectric that is non-magnetic. Copper itself is only weakly paramagnetic so one is justified in taking the free space value for the magnetic permeability everywhere in this system. The magnetic permeability of free space is $\mu_0 = 4\pi(10^{-7})$ Henry per meter. Fig. 1 is a cross-sectional view of the system.

Fig. 1 displays a cross-sectional view of the system where one must state at the outset that the conditions to be described are those that exist in the interior of an extended cable and do not include effects that occur near the terminations. Also, the overall length of the cable is small compared with the wavelength of the highest frequency component to be considered here. The outer and inner conductors are depicted as shaded in the figure. The circle with tangential arrows represents a typical line of magnetic flux in the region between the conductors. The direction of the magnetic field at this particular radial distance has a sense as indicated by the arrows when the signal current in the inner conductor is directed away from the reader while the return signal current in the outer conductor is directed toward the reader. No other currents are assumed to be present at this point. There is also a magnetic field in the interior of the center conductor having similar directional properties that has zero value on the z-axis but grows as one moves off of the z-axis and moves toward the periphery of the center conductor. At the low end of the audio band, the current is distributed uniformly over the cross-section of each conductor. Ampere's law now governs

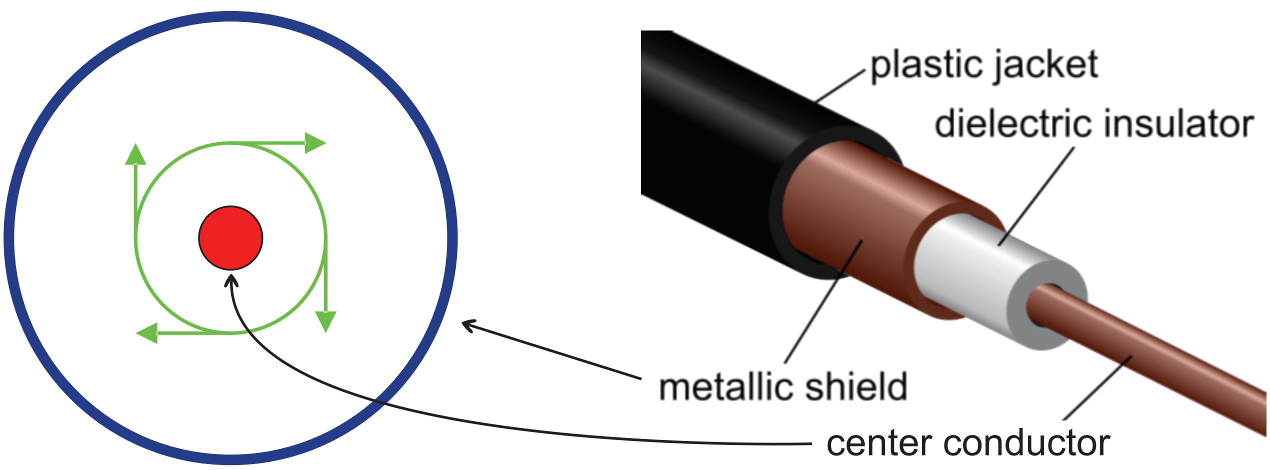

Figure 1. System Geometry.

the calculation of the magnetic induction $\vec{B}$ everywhere for all radial distances from the z-axis. $\vec{B}$ is a vector quantity whose directional properties have just been indicated. The magnitude of this vector will be written as B. For the cylindrical symmetry of the present case, Ampere's law tells one that at all points on a circle of radius r centered on the z-axis the magnitude of the magnetic induction is a constant and that $(B)(2\pi r)$ is equal to μ_0 multiplied by the total current directed through the circle. If the radius of the circle is less than that of the center conductor then the fraction of the total current will be $(i)(\pi r^2 / \pi a^2)$ and the magnetic induction will have a magnitude given by

$$B = \frac{\mu_0 i r}{2\pi a^2}$$

In this equation, i is the instantaneous current, r is the radial distance from the z-axis, and, a, is the radius of the center conductor. This equation applies over the interval $0 \leq r \leq a$. It should be apparent that in this region B grows linearly as r increases up to the point where r is equal to a, where B reaches its maximum value

$$B = \frac{\mu_0 i}{2\pi a}$$

It is important to note that the magnetic field anywhere in the center conductor depends only on the current existing in the center conductor alone. Now consider a circle whose radius exceeds a, but is less than b. Now the region between the conductors is under examination. For this circle, the total current in the center conductor passes through hence

$$B = \frac{\mu_0 i}{2\pi r}$$

This expression is valid for the interval $a \leq r \leq b$. Again, the magnetic field in this region depends only on the current in the center conductor. It should be apparent now that the magnitude of B varies inversely with respect to the radial distance in this region until one reaches the inner radius of the outer conductor at which point B has the value

$$B = \frac{\mu_0 i}{2\pi b}$$

Finally, let the radius of the circle exceed b such that it is located in the interior of the outer conductor. The net current piercing the plane of the circle now is the total current of the center conductor less the fraction of the oppositely directed current of the outer conductor. This net current value is given by

$$i\left[\frac{c^2 - r^2}{c^2 - b^2}\right]$$

An application of Ampere's law yields an expression for B in this region that in turn is given by

$$B = \frac{\mu_0 i}{2\pi r}\left[\frac{c^2 - r^2}{c^2 - b^2}\right]$$

This expression is valid for $b \leq r \leq c$. Note however when $r = c$, at the outer surface of the outer conductor, the value of B has become zero. For all radial distances larger than c, B remains at zero as the net current passing through a circle of such radius is zero and thus there is no external magnetic field. How does this situation change with frequency? As the frequency is increased the skin effect will come into play such that the current distribution within the conductors will no longer be uniform and the fields interior to each conductor will be modified whereas the magnetic field in the region between the conductors will be unchanged. The exact frequency at which this becomes important depends on the cable dimensions, but is typically between 10^4 and 10^5 Hz. At frequencies well beyond the audio band all of the current supported by the center conductor will be restricted to a narrow region near the outer surface of this conductor. A similar statement will be true of the outer conductor except that the narrow region will be restricted to the inner surface of this conductor. The mechanisms that produce these distributions are the eddy currents induced in the conductors by the rapidly changing magnetic fields interior to the conductors themselves. The interior magnetic field grows with radius in the center conductor so the eddy currents reinforce near the outer surface and cancel at the interior of this conductor. For the outer conductor, just the opposite is true. The magnetic field decreases with an increase of radius and the eddy currents reinforce at the smaller radius and cancel at the exterior. As the frequency continues to increase these regions will become increasingly smaller. The upshot is two fold. There will be practically no magnetic field in the interior of either conductor and the resistance of each conductor will be greatly larger than the dc resistance and will continue to increase as the frequency increases.

The fundamental approach to the calculation of the self-inductance of such a system involves first the calculation of the energy stored in the magnetic field. The magnetic energy per unit volume or the magnetic energy density is given by

$$u_m = \frac{B^2}{2\mu_0}$$

In order to calculate the total stored magnetic energy, it is necessary to integrate the energy density throughout the entire volume in which the magnetic field exists. The calculation must be carried out in three steps: the inner conductor, the region between conductors, and the outer conductor. As the total volume involved hinges on the length, an arbitrary length Δz along the z-axis will be considered when constructing the differential of volume. The differential of volume is then $2\pi r dr \Delta z$. The magnetic energy stored in the center conductor at any instant is then given by

$$W_{in} = \int_0^a \frac{1}{2}\mu_0 \frac{i^2 r^2}{4\pi^2 a^4} 2\pi r dr \Delta z = \frac{\mu_0 i^2 \Delta z}{16\pi}$$

The calculation for the region between the conductors proceeds in the same fashion except now one must employ the energy density associated with the magnetic field in this region. The magnetic energy stored in the intermediate region at any instant is given by

$$W_{int} = \int_a^b \frac{\mu_0 i^2}{8\pi^2 r^2} 2\pi r dr \Delta z = \frac{\mu_0 i^2}{4\pi} \Delta z \log_e\left(\frac{b}{a}\right)$$

Lastly, the magnetic energy stored in the outer conductor is calculated from

$$W_{out} = \int_b^c \frac{\mu_0 i^2 \left[c^2 - r^2\right]^2}{8\pi^2 \left[c^2 - b^2\right]^2 r^2} 2\pi r dr \Delta z$$

The evaluation of this integral leads to the result

$$W_{out} = \frac{\mu_0 i^2 \Delta z}{4\pi\left[c^2 - b^2\right]^2}\left(c^4 \log_e\left(\frac{c}{b}\right) - \frac{3}{4}c^4 + c^2 b^2 - \frac{1}{4}b^4\right)$$

The total system stored magnetic energy at any instant is the sum of these three results. The relationship between the total system self-inductance and this stored energy is

$$W_{tot} = \frac{1}{2} L i^2$$

where L is the coefficient of self-inductance and i is the instantaneous current. Electromagnetic theory itself does not require the partitioning of this total inductance to various parts of the system though it is often found convenient to do so particularly in this system where there is no external magnetic field associated with the signal current. In determining the individual inductance associated with each of the three regions one equates the stored magnetic energy in each region to one-half the respective inductance multiplied by the current squared as given in the general relation. The interior inductance of the center conductor at low frequencies is then

$$L_{in} = \frac{\mu_0}{8\pi} \Delta z$$

The inductance per unit length for this region of course is this expression divided by Δz. This is appropriately called the inner inductance of the center conductor. At a frequency for which there is a fully developed skin effect, skin depth much smaller than conductor radius, this inductance becomes zero, as there is no longer an interior magnetic field in the center conductor.

The inductance in the intermediate region between center conductor and outer conductor is calculated in the same fashion except now one employs the magnetic energy stored in this region. The result is

$$L_{int} = \frac{\mu_0}{2\pi} \log_e\left(\frac{b}{a}\right) \Delta z$$

This is by far the principal inductance in the system and is the one quoted when discussing the radio frequency range. Many authors refer to this value as being the outer inductance of the center conductor.

Lastly, the same type of calculation applied to the outer conductor yields

$$L_{out} = \frac{\mu_0 \Delta z}{2\pi\left[c^2 - b^2\right]^2}\left(c^4 \log_e\left(\frac{c}{b}\right) - \frac{3}{4}c^4 + c^2 b^2 - \frac{1}{4}b^4\right)$$

This is the inductance associated with the outer conductor at low frequencies. This as well as the inner inductance of the center conductor diminish as the skin effect begins to develop and of course vanish for a fully developed skin effect.

The conclusions to drawn from this calculation with regard to the outer conductor that supports only the oppositely directed return current are two-fold. Firstly here exists no magnetic field in the interval $b \geq r \geq 0$ as a result of the return current. As a result, there can be no voltage induced anywhere in this same interval when the return current changes with time. At a radius greater than b, the return current in the outer conductor begins to reduce the strength of the magnetic field produced by the current in the center conductor such that by the time the radius c is reached, the magnetic field becomes zero. Secondly, there is an inductance associated with the outer conductor.

Now, consider the case where the desired signal source is not driving the cable in the conventional manner, but some independently sourced noise current exists in the outer conductor alone and the center conductor is unconnected. This current will be designated as I. This current is considered to be time dependent and can change randomly from moment to moment. For the low frequency components of this noise current, the current will be uniformly distributed over the cross-section of the outer conductor. Refer again to Fig. 1 and take the positive direction of the current to be toward the reader. Again from the cylindrical symmetry of the problem the magnetic field is most readily determined by applying Ampere's law. Upon constructing a circle of radius r centered on the z-axis, for $0 \leq r \leq b$, the current passing through this circle is zero so there is no magnetic field in this region produced by the noise current in the outer conductor. Now for $b \leq r \leq c$, the current passing through the circle is

$$I\frac{\left(r^2 - b^2\right)}{\left(c^2 - b^2\right)}$$

and the magnetic field in the interior of the outer conductor due solely to this current is then

$$B = \frac{\mu_0 I}{2\pi r}\frac{\left(r^2 - b^2\right)}{\left(c^2 - b^2\right)}$$

This magnetic field produces an energy density given by

$$\frac{\mu_0 I^2}{8\pi r^2}\frac{\left[r^2 - b^2\right]}{\left[c^2 - b^2\right]}$$

If now the inductance of the outer conductor is calculated by the previously employed technique it will be found to be

$$\frac{\mu_0 \Delta z}{2\pi\left[c^2 - b^2\right]^2}\left[b^4 \log_e\left(\frac{c}{b}\right) - b^2c^2 + \frac{c^4}{4} + \frac{3b^4}{4}\right]$$

This result is larger now because the magnetic field is that of the noise current in the outer conductor alone. Notice also that now there is an external field. In fact at the outer surface of the outer conductor the magnetic field has a value of

$$\frac{\mu_0 I}{2\pi c}$$

Also, of course there is no magnetic field for $r \le b$. The example, of course, is incomplete, as a return path for the current must exist somewhere outside of the cable. The example can't be completed exactly without a detailed knowledge of the geometry of the current return path. Current in the return path may indeed produce a magnetic field in the interior of the cable. One can safely conclude, however, that the current existing in the outer conductor will see both a series resistance and inductance. Incidentally, in this instance the skin effect here will force the current to the outer surface of the outer conductor at high frequencies as the magnetic field in this conductor increases with radial distance.

With the foregoing as a background, it is now possible to look at some situations that involve both a desirable signal current as well as common mode current. Denote the desirable signal current by the letter i, and the common mode current by the letter I. Both of these symbols represent the instantaneous values of the respective currents. These currents co-exist on the coaxial cable of Fig. 1. The two currents i and I are incoherent and thus have random relative phases. As a result they must be summed quadratically rather than linearly when calculating the energies or average power associated with combined currents. The quadratic sum is

$$\sqrt{i^2 + I^2}$$

Now both currents exist on the center conductor. Let the positive direction of each current be away from the reader in Fig. 1. Similarly, both currents exist in the outer conductor and constitute the return current. The geometry and method of calculation follow that of our original example except now one must employ the combined current. The magnetic field in the interval $0 \le r \le b$ again depends solely on the combined current in the center conductor. There is indeed common impedance coupling between the noise and signal from both the resistance of the center conductor as well as the inductance associated with the center conductor. Additionally, there is common impedance coupling between the noise and signal from both the resistance and inductance of the outer conductor. The reactive voltage component of the common impedance coupling on the outer conductor differs from that on the center conductor because the inductances of the two conductors are greatly different. Remember also that the time changing current in the outer conductor does not induce any voltage in the interval $b \ge r \ge 0$. This example constitutes perhaps a worst case of noise pollution of the desired signal and is a strong argument for employing only balanced lines in dealing with audio signals.

For the final example, an insulated, tightly twisted pair of conductors replaces the center conductor of Fig. 1, and the outer conductor makes a snug fit with the twisted pair. The desirable signal current exists only in the twisted pair with one wire supporting the send signal current while the other wire supports the return signal current. Additionally, on each of the members of the twisted pair there exist equal noise currents of I / 2 flowing in the send direction while on the outer conductor there exists only the return noise current I. This would be the situation existing in an accurately balanced system with a common mode voltage source. As far as the noise current is concerned, this geometry parts from cylindrical symmetry in only a minor way so the interior field produced by the noise current follows that of the coaxial cable example. The very close proximity of the signal send and return currents produces hardly any magnetic field outside of a circle that just contains both conductors. The magnetic field outside the containment circle is thus principally that of the noise current except for a very weak or vanishing noise current. The differential voltage produced between the members of the twisted pair as a result of the changing magnetic field generated by the noise current is minimized by the wire position transpositions introduced by the twists. It is essential that the twists be geometrically uniform in order to produce maximum cancellation.

A few final comments are in order. Magnetic fields for straight single conductors or for closed loop circuits are often calculated by means of the Biot-Savart law in which integration is performed along the length of the current carrying conductor. Such integrations can be quite tedious in the general case. Ampere's law is satisfied for those situations as well though it might not be useful by itself to make the calculation because of geometrical complexities. The governing equations for electromagnetic fields are Maxwell's equations. These equations may be written in both differential and integral form. The differential form is listed immediately below where the arrow overbars designate a vector quantity.

$$\nabla x \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \bullet \vec{D} = \rho$$

$$\nabla x \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \bullet \vec{B} = 0$$

These basic equations are further supported by what are called the constitutive relations that for linear media are stated as the following.

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{J} = \sigma \vec{E}$$

In these equations μ is the magnetic permeability of the medium in question expressed in Henries per meter (in non-magnetic material or free space this is called μ_0) and ε is the electric permittivity of the medium (in free space this is called ε_0). $\vec{B}$ is the magnetic induction measured in Tesla, $\vec{H}$ is the magnetic intensity measured in Ampere per meter, $\vec{E}$ is the electric field strength measured in volt per meter, $\vec{D}$ is the electric displacement measured in Coulomb per meter^2, ρ is the free electric charge density measured in Coulomb per meter^3, $\vec{J}$ is the current density measured in Ampere per meter^2, and σ is the electrical conductivity measured in Siemen per meter.

The first of Maxwell's equations is an expression of Faraday's law of electromagnetic induction; the second expresses the existence of electric monopoles that is one can have a quantity of isolated positive or negative charge. The third is Ampere's law as amended by Maxwell to include the displacement current density. The displacement current density is the second term on the right hand side of the third of Maxwell's equations. It would be significant in the coaxial cables that have been discussed previously only at extremely high frequencies and hence has not entered into the example calculations. The fourth of Maxwell's equations expresses the fact that up until this point in time no one has observed an isolated magnetic pole or magnetic monopole. The third of the constitutive equations is the microscopic statement of Ohm's law. Now if one writes the third of Maxwell's equations in the integral form while neglecting the displacement current density what appears will be Ampere's law as it has been applied in all of the previous examples.

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 \int \vec{J} \bullet d\vec{S}$$

This is particularly easy to apply in problems possessing cylindrical symmetry because the magnetic field has a fixed magnitude at a fixed radius of the path encircling the current and its direction is tangential to the circle so the line integral on the left becomes simply B2πr and the surface integral on the right is just the net current passing through the circle multiplied by μ_0. *ep*