Gauss's Law and the Electrostatic Field

The concept of the electric field may be hazy to some readers so we will begin by reviewing some fundamentals. I am assuming that everyone is familiar with the fact that electric charge exists in two fundamental forms termed positive and negative as originally suggested by Benjamin Franklin and that physical materials can be distinguished electrically as being either conductors, insulators, semiconductors, or very rarely superconductors.

As early as 1785 Charles Augustin Coulomb had performed a series of experiments that led him to what is now called Coulomb's Law. This law states that the mutual electrical force of attraction or repulsion between two electrically charged particles when at rest is directly proportional to the product of the amount of the individual charges and is inversely proportional to the square of the distance between the locations of the particles. Furthermore, similarly charged particles experience a repulsive force whereas oppositely charged particles experience an attractive force with the line of action of the force being that of a line drawn connecting the locations of the two particles. When stated as an equation employing modern terminology and SI units for charged particles located in a vacuum Coulomb's Law becomes

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}.$$

In this equation, F is the mutual force experienced by each charged particle. When the individual charges have the same algebraic sign F will be positive. This means that the forces are repulsive acting to increase the distance between the charges. When the charges have opposite algebraic signs, F will be negative. This means that the forces are attractive acting to decrease the distance between charges. The charges are measured in Coulombs, r is measured in meters, and ε_0 is $8.85(10^{-12})$ Coulomb^{2}Newton^{-1}meter^{-2} for charges located in a vacuum and with negligible change for charges located in air. The force description is summarized in Fig. 1.

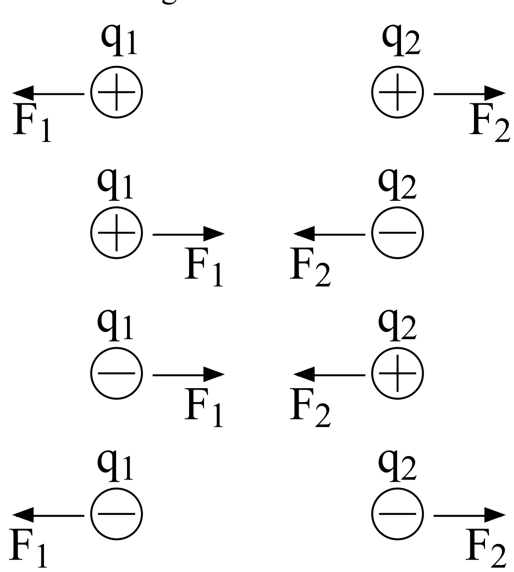

Figure 1. Examples of Coulomb's law.

In all instances of Fig.1 the force experienced by the charge q_1 is F_1 while that experienced by q_2 is F_2. It should be noted that in each case F_1 is oppositely directed to F_2 in agreement with Newton's third law. The forces depicted in Fig.1 are electrostatic forces as the charges are at rest. If the charges had been in relative motion magnetic forces could also come into play. In 1831 as a result of his studies of magnetism, Faraday introduced the concept of a field of force as a way of visualizing the interaction between magnets and current carrying conductors. Later on Maxwell provided a quantitative mathematical foundation for Faraday's field concept as applied to both magnetism as well as electricity. In the electric field concept, the conditions in the space surrounding an electric charge are altered by the mere presence of the charge. Let the quantity of positive charge in Fig. 2 be of an appreciable amount q_1 that is fixed in position and let us explore the conditions in the space surrounding the location of q_1 with the aid of another amount of movable positive test charge q_2. For each point at which we place q_2 with q_2 at rest we measure both the direction and magnitude of the force experienced by q_2. We then divide the magnitude of the measured force at each location by the size of q_2. The pattern depicted in Fig. 2 summarizes our results as viewed in any plane centered on q_1. When viewing Fig. 2, the reader must imagine that the lines in the figure extend indefinitely far in the directions indicated by the arrows.

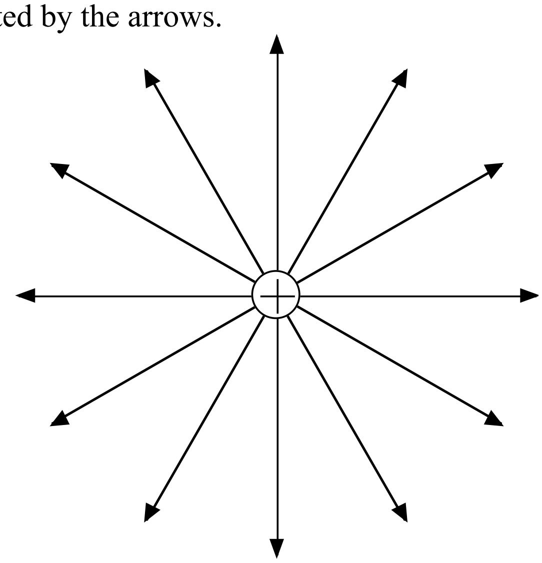

Figure 2. Electrostatic field of a positive monopole.

This structure is called a positive static electric monopole. In three dimensions with q_1 represented by a small sphere the pattern would be that of a fixed number of uniformly distributed radii diverging from the center of the sphere. This pattern indicates not only the direction but can also represent the magnitude of the electrical force per unit charge in the space surrounding the source charge q_1 provided that we let the number of diverging lines be proportional to the size of q_1. In order to understand this, imagine that we enclose q_1 by a second larger spherical surface whose radius is r and with the second sphere being centered on the location of q_1. When q_2 is placed anywhere on the second sphere's surface it will experience a repulsive force $F_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$.

The electric field strength E at the location of q_2 in this simple case is obtained by dividing F_2 by q_2 resulting in

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}.$$

The expression for the magnitude of E given immediately above is exactly correct for the chosen simple example. Returning to the simple example, if now we multiply E by the surface area of the surrounding sphere whose radius is r we will obtain a quantity called the flux of the electric field through the surrounding surface. With the flux of the electric field being denoted as **Φ** and the surface area as S then

$$\Phi = ES = E4\pi r^2 = \frac{q_1}{\varepsilon_0}.$$

The conclusion in this simple case is if the number of lines drawn emanating radially from the charged particle is proportional to the charge of the particle, with the constant of proportionality being

$$\frac{1}{\varepsilon_0},$$

then the flux density or flux per unit area at any point on the surface of the surrounding sphere is equal to the strength of the electric field at that point. The simplicity of this case can be misleading. Henceforth, we will represent vector quantities by bold-faced type and the magnitude of such quantities by normal typeface. The electric field is a vector quantity and its direction must always be taken into account. Mathematically, any infinitesimally small area located on the surface of the surrounding sphere is also represented by a vector quantity. This differential of area written as a vector would appear as d**S**. The direction of this element of area vector is along the outwardly pointing perpendicular to the closed surface. Unlike scalar quantities, that is, quantities that have only a magnitude and no direction, vector quantities can be multiplied in two uniquely different ways called the dot or scalar product and the cross or vector product. In this instance we are interested in the scalar product written as **E**•d**S**. The magnitude of this product is the magnitude of the electric vector multiplied by the magnitude of the element of area vector times the cosine of the angle between the directions of the two vectors written as EdScos(θ). In the example given above **E** is everywhere normal to the surface of the surrounding sphere and of constant magnitude and the same can be said of d**S** so the angle between the two vectors at all points on the surface will be zero and the cosine of the angle will be unity. The correct mathematical statement of our calculation would then appear as

$$\Phi = \oint \mathbf{E} \bullet d\mathbf{S} = \frac{q_1}{\varepsilon_0}$$

where the circle on the integral sign means that the integration is carried out on a closed surface. In this simple case, the integration is quite easy because the magnitude of the electric field is constant everywhere on the surface and its direction is everywhere parallel to the differential of area so the value of the integral is just ES. Suppose, however, the center of the surrounding sphere had been unwisely chosen to be slightly offset from the location of the charge. In such an instance a very difficult integration would be required because **E** would be changing in both magnitude and direction from point to point on the surface of the offset spherical surface. However, the same result would ultimately be obtained because the same number of lines of flux still originates from q_1 and this flux ultimately passes out through the offset enclosing surface. Our equation for the electric flux is a special case of a general law called Gauss's Law that states

$$\oint \mathbf{E} \bullet d\mathbf{S} = \int \frac{\rho}{\varepsilon_0}\, dv.$$

This law states the net flux of the electric field through any closed surface is equal to the density of all charges divided by the electric permittivity of a vacuum integrated throughout the volume defined by the closed surface. In many cases the statement can be abbreviated as simply

$$\Phi = \frac{q}{\varepsilon_0}$$

where Φ is the net flux through the closed surface and q is the net charge contained within the closed surface. A positive value for Φ corresponds to an outwardly directed field and a net positive charge. A negative value for Φ would correspond to an inwardly directed field and a net negative charge. Both the flux and the charge are scalar quantities. They behave as simple algebraic quantities but have no direction. Fig. 3 is a hand drawn sketch of the electric field in a plane containing two equal positive charges. Note that in Fig. 3 the total flux is doubled if a single surface encloses both charges as compared with only one charge being enclosed. Fig. 4 is a hand drawn sketch of the electric field in a plane containing both a positive charge as well as a negative charge of equal absolute magnitude. This structure is called a static electric dipole. Note that here the net flux through any surface that encloses both charges is zero.

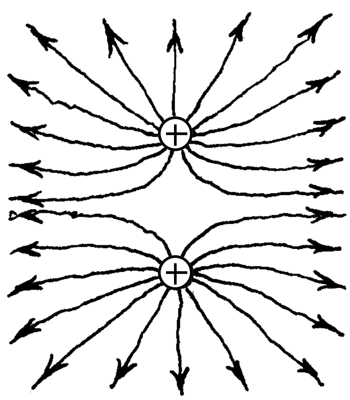

Figure 3. Two equal positive charges.

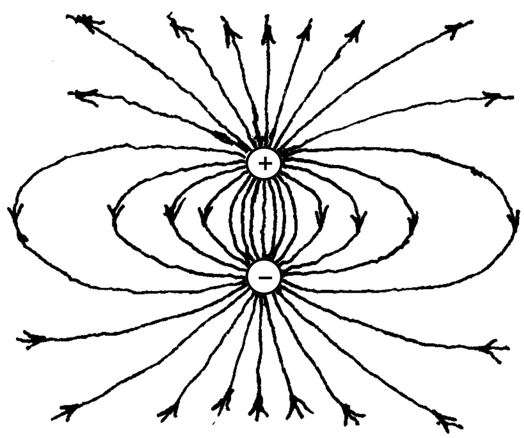

Figure 4. Electric dipole.

An electric dipole is characterized by what is called its dipole moment. The dipole moment is the product of the distance d between the individual charges constituting the dipole and the size of the positive charge. If r is taken to be the distance from the center of the dipole, then when r is much greater than d, the electric field of the static electric dipole diminishes as r^{-3} rather than as r^{-2} as is true for the electric monopole. In the general case where the origin of the electric field may be several discrete charges or even a continuous distribution of charge either on a surface or throughout a volume all of which are at rest, the electric field is defined to be the value of

$$\frac{\mathbf{F}}{q_t}$$

where **F** is the net electric force exerted by the distribution on a small test charge q_t. The unit of **E** is a Newton per Coulomb, which is the same as a volt per meter. The equation

$$\oint \mathbf{E} \bullet d\mathbf{S} = \int \frac{\rho}{\varepsilon_0}\, dv,$$

is the integral form of one of Maxwell's famous equations. We may convert it to the differential form by employing a general theorem governing vector fields known as the divergence theorem that states

$$\oint \mathbf{E} \bullet d\mathbf{S} = \int (\nabla \bullet \mathbf{E}) dv.$$

This allows us to write

$$\int (\nabla \bullet \mathbf{E}) dv = \int \frac{\rho}{\varepsilon_0}\, dv.$$

Finally, since the size of the volume of integration is arbitrary, the statement holds for any volume, which can only be true if the integrands are equal. Therefore,

$$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

Now we will attempt to apply what we have learned so far to a real physical object. Suppose we have a long, straight copper wire and an equally long hollow copper cylinder having a thin wall. The wire is along the central axis of the cylinder so that we now have a coaxial pair of conductors with each conductor being initially uncharged. Copper is an excellent conductor of electricity and an isolated atom of copper has 29 electrons each of which is strongly bound to the nucleus of the atom and resides in a discrete energy level. This situation changes when something of the order of Avogadro's number of copper atoms is brought together to form a metallic solid of copper wire or cylinder. The interaction of the copper atoms with their many neighbors causes the discrete energy levels of the electrons farthest from the nucleus to be smeared into energy bands. The outermost or most energetic band is called the conduction band. Any electrons in this band though attached to the structure as a whole are not attached to a particular nucleus and can move under the influence of an applied electric field throughout the extent of the entire conductor. The next lower band in terms of overall energy is called the valence band and for copper as well as other metallic conductors the valence band and the conduction band have overlapping ranges in energy. The mobile charges responsible for conduction in metallic solids are of course electrons. Contrast this with the case of insulators where the valence band is completely occupied with electrons and there exists a large forbidden energy gap above the valence band before reaching the empty conduction band. Even the best insulators can conduct weakly due in part to the presence of impurities but also because of surface contamination and environmental influences. Gases particularly at low pressures can be made to conduct and the mobile charges are both positive and negative ions as well as free electrons. Charges of opposite signs in conducting gases move in opposite directions under the influence of the local electric field. Conducting liquids are called electrolytes and here again both positive and negative ions participate in the conduction process.

Fig. 5 is a description of a short length *l* of coaxial cable that is located near the midpoint of a long cable having the same structure. The left portion of the figure depicts a sectional view along this length of the cable while the right portion is an end view of the cable. The natural coordinates to be employed here are cylindrical coordinates and the figure facilitates the determination of the relationship between cylindrical and Cartesian coordinates of an arbitrary spatial point in or on the cable. In cylindrical coordinates the location of an arbitrary point is expressed in terms of r, ϕ, and z where r and ϕ are the radial and angle coordinates in a plane perpendicular to the cylinder axis at the point z. From Fig. 5 it is seen that the relationships between the Cartesian and cylindrical coordinates are

$$x = r\cos(\phi)$$
$$y = r\sin(\phi)\ .$$
$$z = z$$

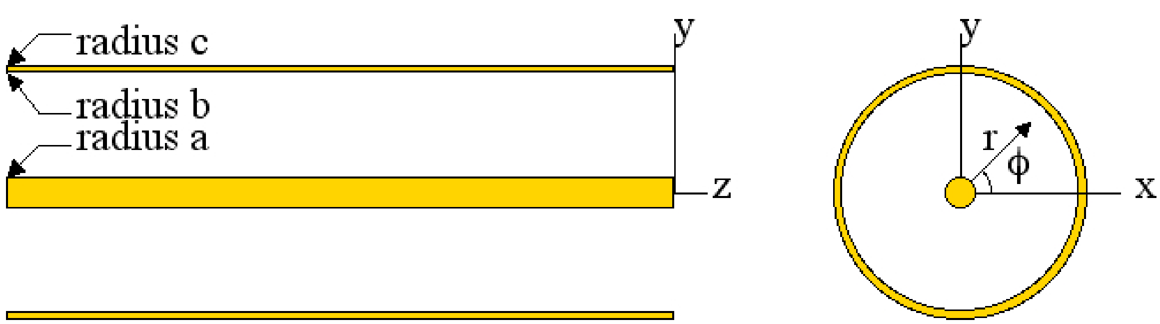

Figure 5. Section of a coaxial cable of length *l*.

Now we want to examine the ultimate consequence of connecting a source of electrical energy between the central conductor and the outer conductor of the coaxial cable. The source in this instance can be quite simple such as an ordinary D cell whose positive terminal is connected in series through a resistance R to the central conductor of the cable while the negative terminal of the cell is connected to the outer conductor. Although we will ultimately investigate the transient behavior that occurs immediately after the connection is made we now are interested in the steady state behavior with the source at one end of the long cable and with nothing connected to the other end. The ultimate action of the source will have been to remove a quantity of negative charge from the central conductor and deposit this same quantity of negative charge on the outer conductor. The immediate question to be answered in detail is what is the charged state of each conductor and where is this charge located when it ultimately comes to rest? Since the source has removed negative charge from the central conductor this conductor now has an excess positive charge and the outer conductor has an excess negative charge of the same size. Let the total quantity of charge moved by the source be designated as –Q. The central coax section having a length *l* bears only a fraction of this charge, which we designate as q. The central conductor of the coax section then has a positive charge q while its outer conductor has a charge –q. If the excess charge associated with the central conductor were located in its interior then an electric field would exist there and charges would be in motion under the influence of this field. In order for this charge to be at rest it must reside on the outer surface of the central conductor and the electric field for which this charge is the source must be everywhere perpendicular to the surface of the central conductor. If the field were not normal to the surface, it would have a tangential component that could exert force on charge at the surface of the conductor and cause that charge to be in motion. The mutual attraction between the negative charge on the outer conductor and the positive charge on the surface of the inner conductor will cause the negative charge associated with the outer conductor to distribute itself on the inner surface of the outer conductor. The electric flux that originates at the outer surface of the inner conductor will terminate at the inner surface of the outer conductor while being perpendicular to the outer conductor's inner surface. Thus when the transferred charge comes to rest a static electric field will exist in the region between the two conductors that has only a radial component. Fig. 6 is an end view of the cable section illustrating the radial lines of the electric field originating at the outer surface of the inner conductor and terminating on the inner surface of the outer conductor.

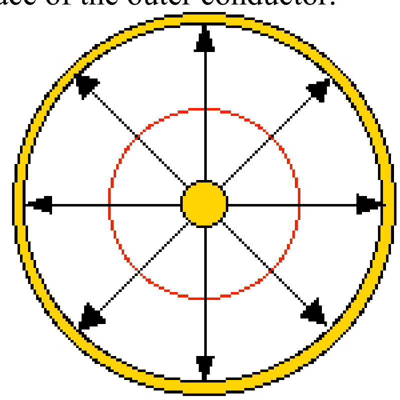

Figure 6. Radial electrostatic field of a charged cable section.

Also shown in the figure is a red circle that represents the end of a closed coaxial cylindrical surface of radius r that extends for a distance *l* along the cable. The lateral surface area of this cylinder is $2\pi r l$ and the electric field is everywhere perpendicular to this lateral surface. Gauss's law tells us that the flux through this surface is

$$\frac{q}{\varepsilon_0}.$$

This allows us to state that the flux density or the electric field strength for a radial distance r in the range $a \le r \le b$ is given by

$$\mathbf{E} = \frac{q}{2\pi\varepsilon_0 l\, r}\hat{r},$$

where $\hat{r}$ is a unit vector in the radial direction. Furthermore, the electric field is 0 in the interior of the central conductor where $r < a$ as well as the interior of the outer conductor where $b < r < c$. In closing this article we need to make one more calculation and make a related important definition. The calculation is that of the scalar potential difference, expressed in volts, between the center conductor and the outer conductor that has arisen because of the present charge distribution. This is denoted as V_{ab}. The proof of the following statement will be supplied at the beginning of the next article in this series.

$$V_{ab} = \int_a^b \mathbf{E} \bullet d\mathbf{r} = \int_a^b \frac{q}{2\pi\varepsilon_0 l} \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 l} \ln\left(\frac{b}{a}\right)$$

In the above equation, the term ln( ) stands for the natural logarithm of the parenthetical quantity. Finally, the electrical capacitance, expressed in farads, of the section of coaxial cable of length l is defined to be the ratio of the transferred charge of the conductors divided by the resulting potential difference between the conductors or

$$C = 2\pi\varepsilon_0 \frac{l}{\ln\left(\frac{b}{a}\right)}.$$